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SELF-SIMILAR PROBLEMS OF THE DYNAMIC BENDING OF INFINITE NONLINEARLY ELASTIC BEAMS

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Obtaining the exact solutions of dynamic bending problems for beams whose material is not subject to Hooke's law, is fraught with great mathematical difficulties. Approximate methods are used in solving such problems. For instance, the dynamic bending of infinite nonlinearly elastic beams is investigated in [1] by using series expansions of the solution in a variable interval. According to [2, 3], in solving the problem the beam is replaced by a chain of stiff sections interconnected by hinges along the length, wherein elastic or plastic elements with characteristics averaged over the length of each section are concentrated.

In this paper an exact solution on the bending of physically nonlinear infinite beams subjected to concentrated effects is obtained. The beam material is subject to a power-law dependence between the curvature and bending moment. The property of self-similarity of the problem [4, 5] is used to obtain this solution.

1. A homogeneous prismatic beam is considered, whose bending is described by the equation

$$\partial^2 M/\partial x^2 + m\partial^2 w/\partial t^2 = q(x,\ t), \tag{1.1}$$

where x is a coordinate measured along the beam, t is the time, w is the deflection, M is the bending moment, m is the linear mass of the beam, and q(x, t) is the linear load.

In order to be able to construct a self-similar solution in describing the dependence between the beam curvature and the bending moment, the simplest relationship containing the minimal number of dimensional quantities must be used. The power-law dependence [6]

$$M = M_0(|\partial^2 w/\partial x^2|)^{\mu}\,\mathrm{sign}\,(\partial^2 w/\partial x^2), \tag{1.2}$$

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satisfies such a condition, where μ is the exponent to be given, and M_0 is a dimensional constant. The vertical lines denote the absolute value, while the symbol sign denotes the sign of the corresponding quantity.

The self-similar solution of (1.1) and (1.2) is sought in the form

$$w = w_* t^{\alpha} \varphi(\xi); \tag{1.3}$$

$$M = M_* t^{\delta} \psi(\xi), \tag{1.4}$$

where α, δ are still unknown exponents, w_*, M_* are dimensional factors, and $\varphi(\xi)$, $\psi(\xi)$ are dimensionless functions of the dimensionless variable ξ.

In the self-similar solution the variable ξ has the form [4]

$$\xi = x/(2bt^{\beta}), \tag{1.5}$$

where β is an exponent, and b is a dimensional constant. The values of β and b are determined below. The factor 2 in the denominator is taken for convenience in the subsequent manipulations.

In order to be able to construct a self-similar solution, the right side of (1.1) should be given in the form

$$q = q_* t^{\omega} f(\xi), \tag{1.6}$$

where q_* is a dimensional constant, ω is an exponent, and $f(\xi)$ is a dimensionless function of the variable ξ.

The values of (1.3), (1.4) are substituted into (1.1) and (1.2) and the dependence (1.6) is taken into account. The dependence of the variable ξ on t and x is taken into account in the differentiation.

We obtain

$$\begin{gathered}\psi'' + mw_* M_*^{-1}(2b)^2 t^{\alpha-2-\delta+2\beta}[\beta^2\xi^2\varphi'' - \beta(2\alpha-\beta-1)\xi\varphi' + \\ + (\alpha-1)\alpha\varphi] = q_* M_*^{-1}(2b)^2 t^{\omega-\delta+2\beta} f(\xi);\end{gathered} \tag{1.7}$$

$$\psi = M_0 M_*^{-1}\left[w_*(2b)^{-2}\right]^{\mu} t^{\mu(\alpha-2\beta)-\delta}\,|\varphi''|^{\mu}\operatorname{sign}\varphi''. \tag{1.8}$$

The primes in these equations indicate differentiation with respect to the variable ξ. Equations (1.7) and (1.8) should not contain t explicitly. Hence, all the exponents of t must be equated to zero

$$\alpha - 2 - \delta + 2\beta = 0,\ \mu(\alpha - 2\beta) - \delta = 0; \tag{1.9}$$

$$\omega - \delta + 2\beta = 0. \tag{1.10}$$

If there is no transverse load q, then the three unknown exponents are just related by the two equations (1.9). The third equation can be obtained from an additional, say boundary, condition.

Let us note that the initial and boundary conditions in the self-similar solution cannot be arbitrary. For instance, if $q(x, t) = 0$, then only one boundary condition, of independent dimensionality, can be taken. The remaining conditions should here have dimensionalities dependent on the dimensionalities of the quantities introduced earlier, or be zero. If $q(x, t) \neq 0$, than all the remaining conditions should have a dependent dimensionality or be zero. Appropriate examples are presented below. The reasoning elucidated predetermines the set of problems which can have self-similar solutions. Here, in particular, are problems on the bending of infinite beams excited kinematically or by forces acting in the section x = 0 under zero initial conditions. Infinite beams have no characteristic linear dimension and are subject to zero boundary conditions at infinity, i.e., have a minimal number of dimensional constants, as is essential for the construction of a self-similar solution.

2. Let us consider the bending of an infinite beam subjected to the force P applied in the section $x = 0$

$$P = P_* t^{\lambda}, \tag{2.1}$$

where P_* is a dimensional factor, λ is a given exponent. The distributed load $q(x, t)$ in (1.1) is set equal to zero.

Expressing the transverse force in terms of (2.1), we obtain

$$\partial M / \partial x = 0.5 P_* t^{\lambda}.$$

Let us substitute (1.4) here

$$\psi'(0) = 0.5 P_* 2b M_*^{-1} t^{\lambda-\delta+\beta}. \tag{2.2}$$

The expression obtained should not depend on the time. This yields a third condition to determine the unknown exponents

$$\lambda - \delta + \beta = 0. \tag{2.3}$$

Solving (1.9) and (2.3) jointly, we obtain

$$\alpha = 2[1 + 2\mu + \lambda(1 + \mu)](1 + 3\mu)^{-1}; \tag{2.4}$$

$$\beta = [2\mu - (1 - \mu)\lambda](1 + 3\mu)^{-1}; \tag{2.5}$$

$$\delta = 2\mu(1 + 2\lambda)(1 + 3\mu)^{-1}. \tag{2.6}$$

Now the value of the coefficient b in (1.5) can be established. The dimensionality

$$[b] = LT^{-\beta}, \tag{2.7}$$

follows from (1.5), where L and T are, respectively, the notation for the dimensionalities of length and time. Let us form a combination of the dimensionalities mentioned from the dimensional parameters

$$M_0, m, P_*. \tag{2.8}$$

Taking account of (2.5), we obtain

$$b = \left(M_0 P_*^{\mu-1} m^{-\mu}\right)^{1/(1+3\mu)}. \tag{2.9}$$

Let us find the coefficients w_* and M_* in (1.3) and (1.4). Their dimensionality is determined by the formulas

$$[w_*] = LT^{-\alpha}, \quad [M_*] = KL^2 T^{-2-\delta}, \tag{2.10}$$

where K is the notation for the dimensionality of mass. Taking account of (2.4) and (2.6), combinations corresponding to these dimensionalities can be formed from the parameters of (2.8). The values of the numerical factors in w_* and M_* can be taken arbitrarily since they can be combined with the still unknown functions $\varphi(\xi)$ and $\psi(\xi)$ or extracted from them. The value of the numerical factors is selected for convenience in the subsequent manipulations. Taking this into account, we take w_* and M_* in the form

$$w_* = 4\left(M_0^{\lambda+2(1-\alpha)} P_*^{-2+\alpha(3+\mu)} m^{-\lambda-\alpha(1+\mu)}\right)^{1/[\lambda(3+\mu)+2(1+\mu)]}; \tag{2.11}$$

$$M_* = P_* b = P_* \left(M_0 P_*^{\mu-1} m^{-\mu}\right)^{1/(1+3\mu)}, \tag{2.12}$$

where α is determined from (2.4).

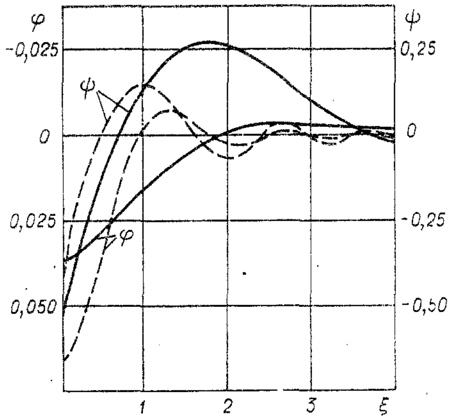

Fig. 1

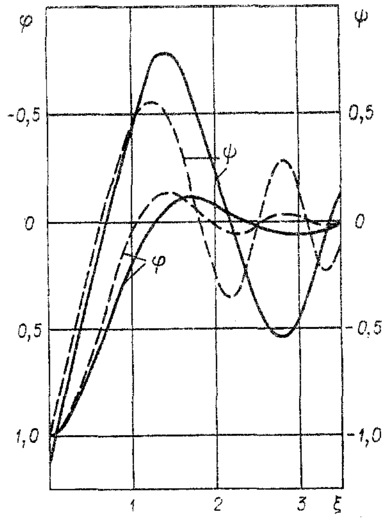

Fig. 2

Substituting (2.11) and (2.12) into (1.7) and (1.8) and taking account of (1.9), we obtain the following system of equations

$$\psi'' + r[\beta^2\xi^2\varphi'' - \beta(2\alpha - \beta - 1)\xi\varphi' + (\alpha - 1)\alpha\varphi] = 0,$$
$$\psi = s|\varphi''|^\mu \operatorname{sign} \varphi''. \tag{2.13}$$

In this system r = 16, s = 1, and the coefficients α and β are related to the given quantities λ and μ by the dependences (2.4) and (2.5).

The system of equations obtained should be integrated under boundary conditions resulting from the boundary conditions of the problem under consideration. If the beam is infinite and subjected to a force in the section x = 0, then the conditions

$$\varphi' = 0,\ \psi' = 1\ (\text{ for }\ \xi = 0),\ \varphi \to 0,\ \varphi' \to 0\ (\text{ for }\ \xi \to \infty) \tag{2.14}$$

should be satisfied.

The first condition results from the angle of rotation being zero under a force, and the second from the expression (2.2) in which the value (2.9) and (2.12) have been substituted. The third condition follows from the deflection and its derivative at infinity being zero.

Numerical computations were performed on the digital computer Mir-2 for the bending of a beam subjected to the force (2.1) which varied according to the unit step law (λ = 0) for μ = 1/3. The system of equations was integrated by the method of reduction to a Cauchy problem [7]. The method is that two conditions (for φ and ψ) are approximately given together with the two known boundary conditions (2.14) on the left end of the beam (ξ = 0) and the Cauchy problem is solved. The magnitude of the boundary conditions for φ and ψ in the left end is corrected by means of the magnitude of non-closure of the solution in the right side of the beam (for sufficiently large values of ξ). The computation is repeated until the function φ and its derivative will be sufficiently small for sufficiently large values of ξ.

Graphs of the functions

$$\varphi = w\,(w_* t^{5/3})^{-1},\ \psi = M\,(M_* t^{1/3})^{-1}$$

are constructed for λ = 0 and μ = 1/3 (solid curves) characterizing the deflection and bending moment in a nonlinear beam in Fig. 1 from the results of computations and analogous graphs

$$\varphi = w\,(w_* t^{3/2})^{-1},\ \psi = M\,(M_* t^{1/2})^{-1}$$

are also constructed for beams with linear elastic properties μ = 1 (dashes).

3. Let us consider the bending of an infinite beam under the kinematic condition

$$v = v_* t^{\nu}, \tag{3.1}$$

given in the section x = 0, where v is the velocity, v_* is a dimensional constant, and ν is the exponent to be given.

As before, the solution is sought in the form of (1.3) and (1.4), the variable ξ has the structure (1.5). In this case (1.7) with right side zero and (1.8), as well as the relationship (1.9) connecting the exponents α, β, δ, μ remain valid. The additional equation for the exponents is obtained from the condition (3.1) upon substitution of the derivative with respect to t from (1.3) therein. Equating the expression

$$\partial w/\partial t = w_* t^{\alpha-1} [\alpha\varphi(\xi) - \beta\xi\varphi'(\xi)]$$

for conditions (3.1) for $\xi = 0$, we obtain

$$\varphi(0) = v_* (w_*\alpha)^{-1} t^{\nu-\alpha+1}. \tag{3.2}$$

The function φ should not depend explicitly on the time in the self-similar solution, consequently

$$\nu - \alpha + 1 = 0. \tag{3.3}$$

Equations (1.9) and (3.3) permit expressing all the exponents in terms of the known quantities μ and ν:

$$\alpha = \nu + 1, \beta = 0.5[2 - (\nu + 1)(1 - \mu)](1 + \mu)^{-1}, \delta = 2\mu\nu(1 + \mu)^{-1}. \tag{3.4}$$

The dimensional factors w_*, M_* and b should be expressed in terms of combinations of the dimensional parameters M_0, m, v_*. Taking account of the dimensionality of (2.7), (2.10), and the relationships (3.4), these constants can be represented in the form of the following combinations

$$b = (M_0 m^{-1} v_*^{\mu-1})^{1/(2+2\mu)}; \tag{3.5}$$

$$w_* = v_*,\ M_* = (M_0 m^{\mu} v_*^{2\mu})^{1/(1+\mu)}. \tag{3.6}$$

Upon substituting these values into (1.7) and (1.8), a system of equations is obtained that has the form of (2.13) in which r = 4, $s = 4^{-\mu}$, and the factors α and β are determined by (3.4).

The system of equations was integrated by the method of reduction to a Cauchy problem [7] for $\nu = 0$ and $\mu = 1/3$ for the boundary conditions

$$\varphi = 1,\ \varphi' = 0 \text{ (for } \xi = 0),\ \varphi \to 0,\ \varphi' \to 0 \text{ (for } \xi \to \infty). \tag{3.7}$$

The first condition in (3.7) results from (3.2) upon substituting the values of w_* (3.6) and α (3.4). The remaining conditions have the same meaning as the analogous conditions in (2.14).

Graphs of the functions

$$\varphi = w(w_* t)^{-1},\ \psi = MM_*^{-1} \tag{3.8}$$

are constructed $\nu = 0$ and $\mu = 1/3$ (solid lines) and for the linear case $\mu = 1$ (dashes) in Fig. 2, from the results of computations. The quantities w_* and M_* in (3.8) are determined by (3.6).

4. An analysis of the results obtained was performed by comparing the solution of the nonlinear problem with the linear case of beam bending studied most ($\mu = 1$). The results of the linear solution can be found in [2, 8]. In this paper the numerical results of the linear problem have also been obtained by a numerical integration of (2.13) for $\mu = 1$ by means of a program composed for the general nonlinear case. The variables used above are inconvenient for the purposes of the investigation. For instance, the magnitude of the variable ξ in

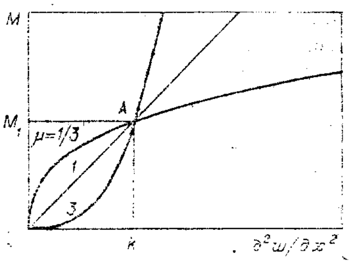

Fig. 3

(1.5) will depend on μ in terms of the quantity (2.9) and will have a different value for the linear and nonlinear cases for all identical initial parameters (2.8) and fixed values of x and t. Moreover, it is difficult to compare stiffness characteristics of beams by using only the dependence (1.2).

Henceforth, another mode of writing (1.2) will be used. A common point A, through which all the curves of power-law dependences of the form (1.2) corresponding to different values of μ will pass, is given in Fig. 3 representing the dependence between the curvature and the moment M. Some bending moment M_1 and curvature k (Fig. 3) will correspond to the point A. The presence of the common point A permits comparing the stiffnesses of beams for different values of μ in identical ranges of variation of the bending moment and curvature. For example, a beam turns out to be stiffer for $\mu = 1/3$ in the range $\partial^2 w/\partial x^2 \in [0, k]$ than a linear beam ($\mu = 1$). The reverse holds for $\partial^2 w/\partial x^2 \in [k; \infty]$.

The factor M_0 is related to the quantities introduced by the formula

$$M_0 = M_1 k^{-\mu} = M_1^{1-\mu} D^{\mu}, \tag{4.1}$$

where D is the bending stiffness of a beam with linear elastic properties, which equals the product of the absolute value of the normal elasticity and the moment of inertia of the transverse section. Using the quantities M_1, D the variable ξ can be converted to the form

$$\xi = 0.5 x_1 \tau_1^{-2\mu/(1+3\mu)}; \tag{4.2}$$

$$\xi = 0.5 x_2 a^{1/(1+\mu)} \tau_2^{-1/2}, \tag{4.3}$$

where $x_1 = xP_* M_1^{-1}$; $\tau_1 = P_*^2 M_1^{-2} \sqrt{Dm}\, t$; $x_2 = x M_1 D^{-1}$; $\tau_2 = v_* M_1 D^{-1} t$; $a = v_* M_1^{-1} \sqrt{Dm}$. Formula (4.2) corresponds to giving the force (2.1) for $\lambda = 0$, and (4.3) corresponds to giving the velocity (3.1) for $\nu = 0$.

Let us reconstruct the graphs in Fig. 1 in new coordinate axes, by laying off the quantities

$$\xi_1 = 0.5 x_1 \tau_1^{-1/2},\ \varphi_* = w P_*^2 D \left(4 M_1^3 \tau_1^{3/2}\right)^{-1},\ \psi_* = M \left(M_1 \tau_1^{1/2}\right)^{-1},$$

respectively, along the abscissa axis and the ordinate axes.
In the new axes (Figs. 4 and 5), the dashed lines corresponding to the linear case will be fixed while the solid curves for the nonlinear case will change in outline as a function of the time. The graphs indicate that the deflection under a force is less, and the bending moment greater in a nonlinear beam for a force effect in the initial period of motion (τ_1 small). Deformations are hence propagated more intensively along the length of a nonlinear beam. As the time increases ($\tau_1 > 32$), the pattern changes. For a nonlinear beam the deformations become more local, i.e., the deflections under a force grow more rapidly and the domain enclosed by the substantial deformations starts to broaden more slowly than in the linear case. The bending moment here grows more slowly than for a linear beam. Such a nature of the deformations can be explained by the fact that for the case $\mu = 1/3 < 1$ under consideration, the stiffness of a nonlinear beam turns out to be greater for small deformations in the initial period than for a linear beam, and its stiffness drops substantially (see Fig. 3) in subsequent times as the deformations grow.

In analyzing the beam deformations for a velocity (3.1) varying as a unit step $\nu = 0$, given in the section x = 0, we note that the bending moment in the section x = 0 grows instantaneously to a certain value of any μ, without changing subsequently. A further analysis could be performed by reworking the graphs in Fig. 2 as follows: the quantity

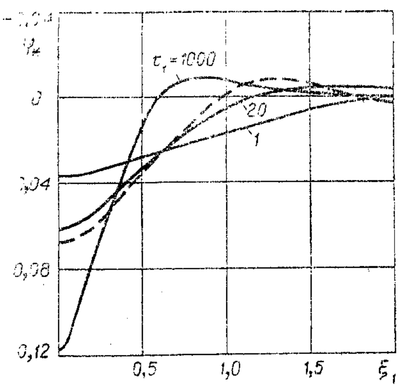

Fig. 4

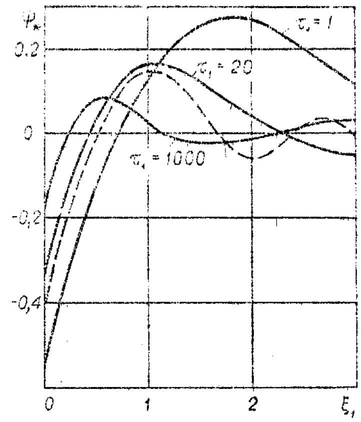

Fig. 5

$$\xi_2 = 0.5 x_2 a^{1/2} \tau_2^{-1/2}, \tag{4.4}$$

is plotted along the abscissa axis, and

$$w\left(D M_1^{-1} \tau_2\right)^{-1} = \varphi, \; M\left(v_* \sqrt{Dm}\right)^{-1} = \psi a^{(\mu-1)/(1+\mu)}, \tag{4.5}$$

along the ordinate axes, where φ and ψ are functions constructed in Fig. 2. The coordinate ξ_2 is related to the old coordinate ξ in (4.3) by the dependence

$$\xi = \xi_2 a^{(1-\mu)/(2+2\mu)}. \tag{4.6}$$

It can be seen that in the linear case ($\mu = 1$), the new graphs of the quantities (4.5) would agree exactly with the dashed lines in Fig. 2. For the nonlinear beam the curve of φ, characterizing the deflection w in (4.5) in the new coordinate system has the same ordinates as the solid line in Fig. 2, however, the scale along the abscissa axis will be altered in conformity with (4.6). The graph of the quantity $\psi a^{(\mu-1)/(1+\mu)}$, characterizing the bending moment (4.5) will have a scale, altered in comparison to Fig. 2, along both the abscissa and ordinate axes for a nonlinear beam, where the nature of the scale change will depend to a greater or lesser degree on the magnitude of the dimensionless parameter a, which can be less or more than one. With the lapse of time not all the lines will change their shape in the coordinate system under consideration.

The parameter a is proportional to the velocity. For high velocities the maximal value of the moment (for x = 0) may turn out to be less in a nonlinear beam with $\mu < 1$ than in the linear case. For limited velocities the relationship between the maximal moments changes to the opposite for $\mu < 1$. For the computed case of a nonlinear beam with $\mu = 1/3$, the boundary value of the parameter a at which the maximal moment in a nonlinear beam becomes equal to the moment in the linear case is determined by the quantity $a = 1.31$.

Let us note that the graphs in Fig. 2 can be considered graphs constructed in the coordinate axes (4.4) and (4.5) for $a = 1$.

Let us examine the possibility of constructing self-similar solutions of infinite beams for other kinds of boundary conditions. Together with the force (2.1), let the concentrated moment

$$M = 2M_2 t^{\varkappa}, \tag{4.7}$$

also act in the section x = 0, where M_2 and $\varkappa$ are, respectively, a given constant and an exponent. This moment is equilibrated by bending moments of the right and left sides of the beam from the section x = 0. Equating half the moment (4.7) to (1.4), we obtain for $\xi = 0$

$$\psi = M_2 M_*^{-1} t^{\varkappa - \delta}, \tag{4.8}$$

from which

$$\varkappa - \delta = 0. \tag{4.9}$$

Condition (4.8) replaces the first condition in (2.14) in the solution of the problem.

The dimensionality of M_2 agrees with the dimensionality of M_*, and can correspondingly be expressed in terms of the dimensionality of the quantities (2.8) introduced earlier. Therefore, a self-similar solution can be constructed by taking account of the dependence of the dimensionality of the parameter M_2 introduced, and in addition, by satisfying (4.9) which sets up a rigorous time dependence of the condition (4.7) in terms of the exponent κ. Other additional nonzero boundary conditions can be introduced analogously.

Let us consider the case $q(x, t) \neq 0$ in (1.1). A self-similar solution can be obtained for zero initial and boundary conditions. Nonzero boundary conditions which are time-dependent, can also be introduced in the section $x = 0$. This dependence is expressed by a power-law function whose exponent is predetermined and will be expressed in terms of given exponents μ and ω. The dimensionality of the constants (factors) in front of this power-law function will correspondingly depend on the dimensionalities of the quantities m, M_0, and q_*.

Therefore, using the properties of self-similarity permits investigation of a sufficiently broad set of questions on the dynamic bending of infinite beams from a nonlinearly elastic material.

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